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$\mathcal{N} = 1^*$ in 5 dimensions: Dijkgraaf-Vafa meets Polchinski-Strassler

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Abstract: One of the powerful techniques to analyze the 5 dimensional Super Yang Mills theory with a massive hypermultiplet ($\mathcal{N} = 1^*$) is provided by the AdS/CFT correspondence. It predicts that, for certain special values of the hypermultiplet mass, this theory develops nonperturbative branches of the moduli space as well as new light degrees of freedom.
We use the higher dimensional generalization of the matrix model/gauge theory correspondence and recover all the prediction of the supergravity analysis. We construct the map between the four dimensional holomorphic superpotential and the five dimensional action and explicitly show that the superpotential is flat along the nonperturbative branches. This is the first instance in which the Dijkgraaf-Vafa method is used to analyze intrinsically higher dimensional phenomena.

Keywords: Field Theories in Higher Dimensions, Duality in Gauge Field Theories, D-branes, AdS-CFT and dS-CFT Correspondence.

JHEP11(2003)001

Contents

1. Introduction

The five dimensional supersymmetric Yang Mills coupled with one massless hypermultiplet has 16 supercharges and is the low energy theory living on a stack of D4 branes. Due to its nonrenormalizability this theory is perturbatively ill-defined. However, its phase structure is more complicated and the high energy regime reveals a six dimensional nature. In a certain energy range the generalization [1] of the AdS/CFT correspondence implies that this theory is dual to supergravity on the near horizon geometry of D4 branes.

Deforming the brane theory by a mass term for the hypermultiplet (and thus obtaining the five dimensional $\mathcal{N} = 1^*$ theory) corresponds on the gravity side to turning on non-normalizable modes of the Ramond-Ramond (RR) 2 form and the Neveu-Schwarz (NSNS) 3 form field strengths. In [2] it was realized that this perturbation can be interpreted as a Melvin twist in the compactification of M5 branes to D4 branes. This observation led to the construction of the exact supergravity dual of the mass-deformed superYang-Mills theory in five dimensions.

JHEP11(2003)001

The supergravity dual has a rather odd feature: for specific values of the five dimensional gauge coupling (such that $A = \frac{1}{4\pi^2} g_5^2 N m_0$ is integer), the D4 branes can polarize [3] into A circular NS5 branes, and the polarization radii are moduli. This implies that for these values of the coupling constant the theory has a "polarization" branch, which intersects the Coulomb branch at points with enhanced gauge symmetry. All vacua on the polarization branch reduce in the classical limit to a *single* vacuum.

We should stress that the existence of polarization branches only at very specific values of the parameters is a stringy phenomenon. Indeed, if $\frac{1}{4\pi^2} g_5^2 N m_0$ were not an integer, one still finds a solution to the supergravity equations of motion. However, this solution has fractional NS5 brane charge and should be discarded since it cannot be a solution of string theory.

To gain an intuitive understanding of the quantum nature of this branch it is perhaps instructive to recall the 4-dimensional $\mathcal{N} = 1^*$ theory [4], to which the 5-dimensional $\mathcal{N} = 1^*$ theory is related by naive dimensional reduction and mass deformation. In that case, to one classical vacuum with unbroken gauge symmetry correspond N quantum vacua, distinguished by the phase of the gaugino condensate. This phase can be interpreted as a (discrete) modulus. In the case we are interested in, to one classical vacuum with unbroken gauge symmetry corresponds a continuum of quantum vacua, distinguished by the polarization radius (which in field theory language represents the expectation value of an operator bilinear in fields).

The purpose of this paper is to recover the predictions of the string dual to the five dimensional $\mathcal{N} = 1^*$ theory using the extension of the matrix model — gauge theory relation to theories in more than four dimensions [5]. The original formulation of this relation [6] allows one to compute holomorphic quantities of a 4 dimensional gauge theory using a matrix model. In the spirit of deconstruction, one can interpret a $4 + d$ dimensional theory as a 4-dimensional one with an infinite set of fields labeled by the remaining d coordinates. After a suitable supersymmetry breaking deformation, which lifts all but a discrete set of points of the moduli space, it is possible to use the original gauge theory/matrix model relation to compute the holomorphic superpotential of this "effective" 4-dimensional theory. Because of the remaining coordinates labeling the fields, the matrix model becomes a d dimensional euclidean bosonic matrix field theory.

In the particular case we are interested in, this is a matrix quantum mechanics. Once its action is found, it is still necessary to solve it and find the five dimensional interpretation of the four dimensional holomorphic quantities which can be computed.

A particular deformation preserving four supercharges of the $\mathcal{N} = 1^*$ theory compactified on a circle was recently studied in [7], where the matrix quantum mechanics corresponding to this theory was identified and solved. We will therefore need only to extract the five dimensional physics from the four dimensional $\mathcal{N} = 1$ holomorphic quantities. As we will see later, the fact that the solution is not known for a generic deformation will not be a problem for identifying the features predicted by the supergravity analysis.

JHEP11(2003)001

We will proceed in section 2 to review the supergravity dual of the 5 dimensional $\mathcal{N} = 1^*$ theory and describe some of its properties. In section 3 we will review the DV description [7] of the compactified version of this theory. We will then proceed in section 4 to discuss the features of the superpotential and then match them with the supergravity predictions in section 5 and section 6.

Various discussions on the five dimensional $\mathcal{N} = 1^*$ theory have appeared in the literature. In particular, a solution for the theory with an SU(2) gauge group compactified on a circle was proposed by Nekrasov [8]. It turns out that the resulting superpotential agrees, up to field redefinitions, with the one computed in [7] using DV techniques. The solution proposed in [8] exhibits a symmetry under certain shifts of the mass of the hypermultiplet. As a byproduct of the supergravity discussion in section 2, we will see that the extension of this symmetry to an SU(N) gauge group corresponds to a symmetry of the dual geometry.

2. The theory and its supergravity dual

As it is well known, the gauge-gravity duality [9] relates field theory living on a large number of D-branes to string theory in their near horizon geometry. Our discussion focuses on a mass deformation of the field theory living on a set of N coincident D4 branes.

The original theory has 16 supercharges and its field content is a 5-dimensional vector multiplet and a hypermultiplet. The five real scalars correspond to the five directions transverse to the branes, and transform in the vector representation of the SO(5) R-symmetry group.

The duality between this maximally supersymmetric gauge theory and string theory was first investigated in [1]. This theory is not conformally invariant, and thus different energy ranges have different weakly coupled descriptions. Since the coupling constant has positive dimension, the field theory is (generically) weakly coupled in the infrared. As one increases the energy, the type IIA supergravity background becomes weakly curved and thus provides an appropriate description of the theory. At even higher energies, the dilaton becomes large enough for the M-theory circle to decompactify and for the 11 dimensional supergravity to become the appropriate description of the theory.

This duality can be perturbed on both sides. On the field theory side, we introduce a mass term for the hypermultiplet and obtain the five dimensional $\mathcal{N} = 1^*$ theory. This theory preserves 8 supercharges and has a Coulomb branch parameterized by the expectation values of operators built out of the vector multiplet scalar. On the supergravity side this mass deformation corresponds to turning on a non-normalizable mode [10] of the RR two form and NS-NS three form field strengths in directions transverse to the branes. This deformation removes the freedom of distributing the branes in four of the five transverse directions, which is the supergravity manifestation of four of the five original Coulomb branch directions being lifted.

Let us now turn to a more detailed description of the supergravity solution. As usual, this description of the theory is valid as long as the string coupling is small and the background geometry is weakly curved. This raises the question of interpreting the various singularities that the solution might develop. We will show that the singularities appearing in the weak coupling regime are physical and have a standard field theoretic interpretation.

2.1 The expected-singularities in the dual geometry, and their physical meaning

The supergravity background dual to the 5 dimensional $\mathcal{N} = 1^*$ theory was investigated in [2], using techniques similar to those employed by Polchinski and Strassler [11] in the analysis of the 4 dimensional $\mathcal{N} = 1^*$ theory. It was also observed that this background can be obtained by longitudinally reducing the 11 dimensional supergravity background sourced by M5 branes with "Melvin twists" on two angular directions. In the absence of the M5 branes, this reduction yields the supersymmetric flux-5-brane of Gutperle and Strominger [12]. Adding the M5 branes gives a type IIA solution which is a superposition of the flux 5-brane and D4 branes.[1] The relation between the gauge theory mass parameter and supergravity twist can be easily established by a first order computation. It turns out (perhaps not surprisingly) that the mass of the $\mathcal{N} = 1^*$ hypermultiplet is proportional to the Melvin twist

$$m = B\,. \tag{2.1}$$

Clearly, in the limit of vanishing mass (vanishing twist) one recovers the D4 brane solution and the corresponding maximally supersymmetric gauge theory.

The exact background dual to the 5 dimensional $N = 1^*$ theory is:

$$\begin{aligned} g_s^{-4/3} e^{4\phi/3} &= Z^{-1/3} + B^2\, Z^{2/3} (\rho_1^2 + \rho_2^2) \equiv \Lambda \\ ds^2_{10} &= \Lambda^{1/2} (Z^{-1/3} dx_{\|}^2 + Z^{2/3} dx_{\perp}^2) - B^2 \Lambda^{-1/2} Z^{4/3} (\rho_1^2 d\phi_1 + \rho_2^2 d\phi_2)^2\,, \\ g_s C_{\phi_1} &= \Lambda^{-1} \rho_1^2 B Z^{2/3}\,, \qquad g_s C_{\phi_2} = \Lambda^{-1} \rho_2^2 B Z^{2/3}\,, \\ g_s F_4 &= *_5 dZ^{-1}\,, \qquad *_5 H_3 = B (\rho_1^2 d\phi_1 + \rho_2^2 d\phi_2) \wedge dZ\,, \end{aligned} \tag{2.2}$$

where ρ_1, ϕ_1 and ρ_2, ϕ_2 are the radial and angular coordinates parameterizing the two transverse 2-planes where the Melvin twists were made, x_7 is the direction corresponding to the scalar in the vector multiplet and $*_5$ is the Hodge dual taken with the flat metric in the directions transverse to the branes.

It is important to note that a D4 brane displaced in the (ρ_1, ϕ_1) or (ρ_2, ϕ_2) plane feels a potential forcing it toward the origin. This potential only vanishes when the Melvin twists vanish, which reflects the fact that in this limit maximal supersymmetry is restored.

The solution (2.2) is not weakly curved everywhere. For example, when all the branes are coincident, the Ricci scalar becomes large in a neighborhood of size $g_5^2 N B^2$ around the origin. The situation does not improve substantially even when the branes are distributed on the Coulomb branch with some density $\rho(y)$. The harmonic function changes to

$$Z = \int \frac{\rho(y) dy}{(\rho_1^2 + \rho_2^2 + (x_7 - y)^2)^{3/2}} \tag{2.3}$$

[1] Similarly to the original flux-brane, this solution preserves 8 supersymmetries [13].

which generically leads to a divergent Ricci scalar at $\rho_1 = \rho_2 = 0$. Even though these singularities come with a zero size horizon (as it is easy to see from the metric (2.2)), it is important to understand whether they are physically meaningful.

A similar singularity exists at the location of the D4 branes in the undeformed solution. In that case the singularity is interpreted as a change in the correct description of the theory: from supergravity away from the branes to the perturbative Yang-Mills theory in 5 dimensions close to them. Given its similarity to the D4 brane case, it is reasonable to expect that the singularity of (2.2) is physical and signifies the presence of a weakly coupled field theory description.

Two criteria help us decide. According to one of them [14], a singularity is physical if the g_{00} component of the metric does not diverge as one approaches it. As one can easily see from (2.2), our singularity clearly passes this test. The other criterion [15] implies that a singularity of a supergravity solution has a field theoretic interpretation if it is possible to construct a family of solutions with the singularity covered by horizons of finite size, which has the original solution as a smooth limit.

To check whether our solution passes the second test, we should attempt to make it near extremal. It turns out that this is not very difficult: using the relation between (2.2) and M5 branes, it is easy to see that the near extremal background is simply the KK reduction of the near-extremal M5 brane background. The dilaton is not modified, and the string frame metric is simply

$$\begin{aligned} ds_{10}^2 = {} & \Lambda^{1/2}\left[Z^{-1/3}f(r)dt^2 + Z^{-1/3}dx^2_{1,2,3,4} + Z^{2/3}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_4^2\right)\right] - \\ & - B^2\Lambda^{-1/2}Z^{4/3}(\rho_1^2 d\phi_1 + \rho_2^2 d\phi_2)^2\,, \end{aligned} \tag{2.4}$$

where $f(r) = 1 - R_H^3/r^3$, R_H is the horizon area and $r^2 = x_7^2 + \rho_1^2 + \rho_2^2$. It is a simple exercise to compute the Einstein frame horizon area of (2.4), and to see that it vanishes in the zero temperature limit.

Before we proceed let us remark a rather odd property of the theory implied by the nonextremal solution (2.4):

$$S = \frac{2\pi^2}{3}R_H^4 Z(R_H)^{1/2} \qquad T_H = \frac{1}{3\sqrt{3}\pi R_H Z(R_H)^{1/2}}\,. \tag{2.5}$$

Thus, for a fixed temperature, the horizon area is independent of the Melvin twist.[2] This implies that the entropy of the dual field theory is independent of the hypermultiplet mass. Though such a behavior might appear strange, it is not unique to this system. A similar phenomenon occurs for a field compactified on a circle with twisted boundary conditions $\psi(0) = \psi(2\pi)e^{i\alpha}$. As one shifts α by 2π the massless states become massive, while some massive states become massless. However $\alpha \to \alpha + 2\pi$ is a symmetry of the spectrum and thus the entropy is invariant under this transformation. We will further discuss the gauge theory implications of this observation in section 5.

[2] The equation (2.5) implies that this holds for all distributions which are twist-independent. It is however not hard to see that the only finite distribution for which (2.4) is a solution is the one with all the branes at the origin of the Coulomb branch.

2.2 The unexpected-brane polarization

Apart from finding the supergravity solutions dual to a generic point on the Coulomb branch of the 5-dimensional $\mathcal{N} = 1^*$ theory, in [2] it was observed (using methods similar to those of [11]) that for very specific values of the hypermultiplet mass (such that $\frac{1}{4\pi^2} g_5^2 m N \in \mathbb{Z}$) the D4 branes can polarize into a circular NS5 brane and moreover, that the radius of polarization is a modulus. For other values of m (such that $\frac{1}{4\pi^2} g_5^2 m N' \in \mathbb{Z}$ with N' being a divisor of N), it is possible for N' of the D4 branes to polarize into NS5 branes, while the other $(N - N')$ branes are free to either move on the Coulomb branch or polarize into more NS5 branes. Thus, if $\frac{4\pi^2}{g_5^2 m} \notin \mathbb{Z}$, there is no polarization branch while, if $\frac{4\pi^2}{g_5^2 m} \in \mathbb{Z}$, the polarization branch intersects the Coulomb branch in points with nonabelian symmetry $\mathrm{SU}(\frac{4\pi^2}{g_5^2 m})$.

This puzzling phenomenon can be explained rather easily by recalling the M-theory origin of the background. Let us pick a uniform, circular array of N M5 branes in one of the (ρ, ϕ) planes. If the Melvin twist accompanying the reduction to 10 dimensions matches the ends of two neighboring branes, the resulting type IIA configuration will be one circular NS5 brane with N units of D4 brane charge. If however the twist does not match brane ends, the descending configuration will not have integer local NS5 charge, and will not be a consistent solution of string theory. In general, only shifts which match brane ends yield consistent IIA solutions with D4 branes inside cylindrical NS5 branes.

The Killing vector along x_{11} is proportional to the Killing vector along ϕ. This implies that the matching of M5 brane ends is independent of the radius. Thus, the D4-NS5 configuration is a solution at any radius, and therefore the potential for the radius of the brane configuration is flat. This reproduces the result of the Polchinski-Strassler-type analysis [2].

To summarize, supergravity predicts a dramatic change in the infrared physics depending on the integrality properties of $A = \frac{1}{4\pi^2} g_5^2 m N$. When the branes are coincident, the solution has a "good" singularity in the infrared, signifying the flow to a weakly coupled gauge theory at low energies. When A is not an integer, one can soften the singularity by distributing the branes on the Coulomb branch, but the singularity never disappears. This implies that infrared physics of these $\mathcal{N} = 1^*$ theories has a weakly coupled field theory description.

When A is integer one can also soften the singularity by going on the polarization branch. The geometry is still given by (2.2), where now the harmonic function Z is sourced by branes on a circle. At a generic point on this branch supergravity is weakly coupled everywhere in the bulk.[3] This implies that for integer A and nonvanishing expectation value of the bilinear operator constructed out of the two complex scalars in the hypermultiplet, the infrared theory remains strongly coupled.

The supergravity solution discussed above predicts that further interesting phenomena occur in the theories with strong infrared dynamics; we will return to them in section 6.

[3]This is easy to see, because on the polarization branch the brane configuration is locally an NS5 brane.

2.3 On a conjectured symmetry of the gauge theory

As briefly mentioned in the introduction, the construction of the supergravity dual of the five dimensional $\mathcal{N} = 1^*$ theory allows us to prove the conjecture [8] that this theory is invariant under shifts of the mass parameter m by a quantity proportional to the inverse of the five dimensional gauge coupling. It is not hard to translate the precise form of the symmetry transformation in supergravity language using the observation that the mass of the hypermultiplet is proportional to the Melvin twist relating the 10 and 11 dimensional geometries (2.1). The shift of the mass parameter conjectured in [8] to be a symmetry of the theory corresponds to changing the Melvin twist by 2π.

A similar invariance in the case of an exactly solvable string theory does exist, as discussed in [16]. By explicitly computing the spectrum it was shown that 2π shifts of the Melvin twist are nontrivial automorphisms both of the Kaluza-Klein tower of supergravity states and of the full string spectrum. It is however difficult to make similar statements about string interactions.

Even though it is not possible to directly extend even the simple spectrum argument to the case we are interested in,[4] it is possible however to reach the same conclusion as in the string theory by using symmetry arguments. The basic observation is that the supergravity fields and their Kaluza-Klein modes form short representations of the supersymmetry algebra, while the massive M-theory modes form long representations. Moreover, 2π shifts of the Melvin twist leave invariant the 11 dimensional supergravity spectrum and the boundary conditions defining the reduction to 10 dimensions. Therefore, these shifts are automorphisms of the 10 dimensional spectrum of supergravity and KK states.

These ingredients are sufficient to establish the shift invariance at the level of holomorphic quantities. From the invariance of the background it follows that the 1-point functions (which are also insensitive to bulk interactions) of operators forming short multiplets are unchanged (up to field redefinitions) by 2π shifts of the Melvin twist. Standard arguments imply that these quantities determine the superpotential via

$$\langle O_n \rangle = \frac{\delta W_{\text{eff}}}{\delta g_n} \qquad W_{\text{tree}} = \sum g_n O_n \, , \tag{2.6}$$

up to g_n-independent terms. Therefore, the superpotentials of theories related by 2π shifts of the mass parameter of the hypermultiplet are the same up to field redefinitions, which proves the conjecture of [8].

It is interesting to note that the shift symmetry of the 5 dimensional $\mathcal{N} = 1^*$ theory holds also at the Kahler potential level. Indeed, due to the extended supersymmetry of the field theory, the Kahler potential and the superpotential are related. Proving this statement from a bulk standpoint is a probably a hard problem given our poor understanding of the quantum M theory. However, this is not unexpected, since in the flat space example discussed in [16], the full string spectrum is symmetric under these shifts.

[4]since we do not have a quantum realization of M-theory

3. Matrix quantum mechanics; the $4d$ superpotential

As we have briefly described in the introduction, the relation between matrix models and 4 dimensional gauge theories can be extended to a relation between d dimensional Euclidean bosonic field theories and $4+d$ dimensional gauge theories.

To do this one formally separates the $4+d$ coordinates into 4 "external" coordinates x and d "internal" coordinates y, and treats each field $\Phi(x,y)$, as an infinite set of 4 dimensional fields labeled by y. The kinetic terms along the y coordinates appear as (super)potential terms in the 4 dimensional theory. This deconstruction procedure [17, 18] can be carried out either at the component level or in superspace and yields a 4 dimensional (super)potential presented as an integral over the d internal coordinates. Quite generally, the resulting 4 dimensional theory has extended supersymmetry. Adding suitable supersymmetry breaking terms it is then possible, using the gauge theory/matrix model relation [6], to compute the holomorphic superpotential in the resulting $\mathcal{N}=1$ theory. The matrix model obtained in this way has an infinite number of fields labeled by the d internal coordinates; it is in fact a d-dimensional Euclidean matrix field theory.

This construction leads to the idea that every d dimensional bosonic matrix field theory describes the sector of some $4+d$ dimensional supersymmetric gauge theory which becomes the holomorphic sector when "deconstructed" to 4 dimensions. However, not all $4+d$ dimensional theories described in this way have $4+d$ dimensional Lorentz invariance. Indeed, it is not hard to see that only superpotentials which are linear in derivatives along the internal directions give regular kinetic terms in $4+d$ dimensions.

In this section we are interested in slightly extending the analysis [7] of the 5 dimensional $\mathcal{N}=1^*$ theory compactified on a circle of radius β. According to the general philosophy described above we should solve a matrix quantum mechanics. It turns out however that the gauge symmetry can be used to set one of the three fields to a constant, makings things somewhat simpler.

To understand the simplification of the matrix quantum mechanics associated to the $\mathcal{N}=1^*$ theory let us first review the case of pure 5-dimensional super-Yang-Mills theory. By compactifying it on a circle the component of the gauge field along the compact direction, A_t, becomes a scalar, and combines with the scalar of the vector multiplet of the five dimensional theory to form the complex scalar Φ_3. Using local gauge transformations, one can set the nonconstant part of A_t to zero. However, one still has large gauge transformations which shift the eigenvalues of A_t by $2\pi/\beta$. Thus, the natural gauge invariant quantity is not A_t but the holonomy of the gauge field along the compact direction, $e^{i\beta A_t}$, which is independent of t. Furthermore, due to $\mathcal{N}=1$ supersymmetry in 4 dimensions the actual variable is the "holonomy" of the superfield Φ_3. Indeed, since the gauge parameter is a chiral superfield, one can use it to set the whole superfield Φ_3 to a constant, up to shifts by $2\pi i/\beta$. The net result of this gauge choice is that the dynamics of Φ_3 is captured by $U \equiv e^{\beta\Phi_3}$. As discussed in [5], this implies that in the case of pure 5-dimensional $\mathcal{N}=1$ gauge theory, the matrix quantum mechanics reduces to a matrix model.

In the case of 5 dimensional $\mathcal{N} = 1^*$ theory things are slightly more complicated due to the presence of the hypermultiplet whose t-dependence cannot be gauged away. Thus, one has to study the full matrix quantum mechanics which nevertheless has one constant field Φ_3. Its action is given by

$$W[\Phi_i] = \mathrm{Tr}[i\Phi_1 D\Phi_2 + m\Phi_1\Phi_2]\,, \tag{3.1}$$

where $D\Phi_2 = \partial_t\Phi_2 + [\Phi_3, \Phi_2]$, and Φ_3 is t-independent.

This superpotential preserves $\mathcal{N} = 2$ in 4 dimensions. To use the Dijkgraaf-Vafa approach [6] it is necessary to deform it by an $\mathcal{N} = 2 \to \mathcal{N} = 1$ breaking term.[5] In principle, any function of Φ_3 which is invariant under $\Phi_3 \to \Phi_3 + 2\pi i/\beta$ (i.e. any function of U) is a suitable deformation. Furthermore, each such function describes a point on the 4-dimensional $\mathcal{N} = 2$ Coulomb branch. Thus, to cover the whole moduli space it would be necessary to use a function with a $\mathcal{O}(N^2)$ arbitrary coefficients. This seems a formidable task, especially because we are interested in finding the precise form of the effective superpotential. We will restrict to the theory studied in [7] — the compactified $\mathcal{N} = 1^*$ deformed by the tree level superpotential $\mu\cosh(\beta\Phi_3)$ — and show that the accessible subspaces of the moduli space already exhibit the nontrivial behavior described in the previous section.

The relevant matrix quantum mechanics was solved using techniques similar to those used to study the Leigh-Strassler deformation of the 4 dimensional $N = 4$ Super Yang Mills theory [20, 19]. The starting point is the partition function:

$$Z = e^F = \int \prod_{1,2,3} [d\Phi_i] \exp\left(-\frac{1}{\beta g_s}\int dt W[\Phi_i]\right), \tag{3.2}$$

where

$$W[\Phi_i] = \mathrm{Tr}[i\Phi_1 D\Phi_2 + m\Phi_1\Phi_2 + \mu(\cosh(\beta\Phi_3) - 1)] \tag{3.3}$$

with $D\Phi_2 = \partial_t\Phi_2 + [\Phi_3, \Phi_2]$. As usual, the effective superpotential is constructed out of the derivatives of the free energy F.

One first integrates out the fields Φ_1 and Φ_2, and obtains a potential which depends only on the holonomy matrix U. Due to its coordinate independence this reduces again the matrix quantum mechanics to a matrix model. After a few rather technical steps [7], one finds:

$$\begin{aligned}
W_{\text{eff}} &= N\frac{\partial F}{\partial S} - 2\pi i\tau S \qquad \tau = \frac{\theta}{2\pi} + \frac{2\pi i}{g_4^2}\\
2\pi i S &= \frac{dh(t)}{dt} = h'(t)\,, \qquad \frac{\partial F}{\partial S} = th'(t) - h(t)\,, \qquad \text{where}\\
h(t) &= \frac{\mu}{\sin\beta m/2}\frac{\theta_1(\beta m/2|t)}{\theta_1'(0|t)}\,,
\end{aligned} \tag{3.4}$$

[5] This procedure, which depending on the deformation selects a point of the $\mathcal{N} = 2$ Coulomb branch as the $\mathcal{N} = 1$ vacuum, was used to derive the known Seiberg-Witten solutions of ordinary four dimensional $\mathcal{N} = 2$ theories.

JHEP11(2003)001

and t is the modular parameter of an auxiliary Riemann surfaces appearing in the solution of the U matrix model. This gives implicitly the superpotential in terms of the glueball superfield S, as well as the effective coupling constant of the remaining U(1) vector multiplet:

$$2\pi i\ \tau_{\text{eff}} = \frac{\partial^2 F}{\partial S^2}\,. \tag{3.5}$$

Knowledge of the effective superpotential leads to knowledge of vacua of the theory. In the limit in which supersymmetry is restored they become vacua of the resulting $\mathcal{N}=2$ theory. Minimizing W_{eff} with respect to t, we obtain

$$h''(t)(tN-\tau)=0 \tag{3.6}$$

which has 2 classes of solutions:

$$t=\frac{\tau}{N} \qquad \text{and} \qquad h''(t)=0\,. \tag{3.7}$$

The distinction between the two types of solutions is crucial. While in the first case the relation between S and t is well-defined, at the points where $h'=0$, this relation becomes degenerate. As we will see shortly, the effective coupling τ_{eff} is not well defined at those points. This suggests a breakdown of the effective description, perhaps accompanied by the appearance of new light degrees of freedom. We will come back to this phenomenon in section 6 and link it with similar features exhibited by the supergravity solution.

Let us consider for now only the "regular" solution $t=\frac{\tau}{N}$. Since a nonvanishing θ angle breaks 5-dimensional Lorentz invariance, we will set it to zero. The vacuum value of the effective superpotential is then

$$W_{\text{eff}} = -\frac{N\mu}{\sin\ \frac{\beta m}{2}}\frac{\theta_1(\frac{\beta m}{2}|\frac{2\pi i}{g_4^2 N})}{\theta_1'(0|\frac{2\pi i}{g_4^2 N})} = -\frac{N\mu}{\sin\ \frac{\beta m}{2}}\frac{\theta_1\left(\frac{\beta m}{2}|\frac{2\pi i\beta}{g_5^2 N}\right)}{\theta_1'\left(0|\frac{2\pi i\beta}{g_5^2 N}\right)}. \tag{3.8}$$

Before proceeding to analyzing in detail the physics of this superpotential it is important to point out that the real part of the mass parameter m in (3.1) corresponds to a Lorentz-breaking term in the 5 dimensional lagrangian, and therefore does not interest us here. The imaginary part of m is the mass parameter of the 5-dimensional hypermultiplet: $m=im_0$. To fix notation, let us state that the 5 dimensional Yang-Mills coupling constant is related to the 4-dimensional one by: $g_5^2 \equiv g_{YM-5}^2 = g_4^2\beta$, where β is the length of the compactified dimension.

In the next section we will analyze the effective superpotential (3.8) and extract the information relevant to the 5 dimensional $\mathcal{N}=1^*$ theory. For this purpose we need to examine the limit $\beta\to\infty$ keeping g_5^2 fixed. Since in this limit the four dimensional coupling is taken to zero, the non-triviality of the result relies on the existence of the infinite set of fields labeled by the fifth dimension. At the same time, in order to recover the full Lorentz invariance we have to restore the eight supercharges, by taking the limit $\mu\to 0$. Since the supergravity description is valid also for finite β, we will explore that case as well. As explained above, the specific choice of supersymmetry breaking term allows us to explore only a subspace of the full moduli space. It turns out that this is enough to expose all the features predicted by the supergravity analysis.

JHEP11(2003)001

4. The Superpotential and its distinguished points

It is relatively easy to see from the superpotential (3.8) that in the space of theories there exists a set of special points distinguished by the integrality properties of $A = \frac{1}{4\pi^2} g_5^2 N m_0$, as predicted by the supergravity analysis.

For ease of notation let us introduce the purely imaginary parameters z and $\tilde{\tau}$ as

$$z = i z_2 \equiv i \frac{\beta m_0}{2} \qquad \tilde{\tau} = i \tilde{\tau}_2 \equiv i \frac{2\pi\beta}{g_5^2 N} \,. \tag{4.1}$$

With this notation the effective superpotential is written as

$$W_{\rm eff} = -\frac{N\mu}{\sin\, z} \frac{\theta_1\,(z|\tilde{\tau})}{\theta_1'\,(0|\tilde{\tau})}. \tag{4.2}$$

The decompactification limit which we are interested in, $\beta \to \infty$, becomes now a double scaling limit

$$z_2, \tilde{\tau}_2 \to \infty \qquad \text{with} \qquad A = \frac{z_2}{\pi \tilde{\tau}_2} = \text{fixed}\,. \tag{4.3}$$

As it is usually the case with expressions involving theta functions, different representations are useful for different purposes. To show that $W_{\rm eff}$ has special properties for integer A, the form (4.2) is not particularly useful, since A does not appear explicitly. An S modular transformation however casts it in a more useful form:

$$W_{\rm eff} = -\frac{N\mu}{\tilde{\tau}_2 \sinh\, z_2} e^{\frac{1}{2}\beta m_0 A} \frac{\theta_1\left(-A| -\frac{1}{\tilde{\tau}}\right)}{\theta_1'\,(0|\tilde{\tau})}. \tag{4.4}$$

Using now the product representation for the theta functions we immediately find that

$$W_{\rm eff} = -\frac{N\mu 2\pi\beta}{g_5^2 N \sinh\, \frac{\beta m_0}{2}} e^{\frac{\beta m_0 A}{2}} \sin(\pi A) \prod_{n=1}^{\infty} (1 + \frac{\sin^2(\pi A)}{\sinh^2(\frac{n g_5^2 N}{2\beta})})\,, \tag{4.5}$$

Thus, for theories for which

$$A \in \mathbb{Z} \tag{4.6}$$

the effective superpotential vanishes for all compactification radii. We should emphasize that the parameter $A = \frac{1}{4\pi^2} g_5^2 N m_0$ has an intrinsic 5-dimensional meaning, as it does not depend on the compactification radius β or on the tree level superpotential breaking the 5 dimensional Lorentz invariance. This implies that, from a 4 dimensional standpoint, the vanishing of the effective superpotential relies entirely on the existence of the continuum of fields labeled by the fifth coordinate. As we have seen in section 2, the integer values of A were also special in the supergravity dual of this theory, being the only ones which allowed brane polarization. We will come back to the relation between the two descriptions in the next section where we discuss the effective superpotential for the physical fields.

Although the product form of the superpotential exposes its zeroes, it does not yield a well defined expansion for $\beta \to \infty$ since all terms in the product (4.5) have contributions of equal magnitude. It turns out that the series presentation of the theta functions is more

JHEP11(2003)001

useful in the analysis of the decompactification limit. From

$$\theta_1(z|\tau) = -i \sum_{n=-\infty}^{\infty} (-)^n q^{(n+1/2)^2} e^{(2n+1)iz}$$

$$\partial_z \theta_1(0|\tau) = 2\sum_{n=0}^{\infty} (-)^n (2n+1) q^{(n+1/2)^2} \tag{4.7}$$

with $q = \exp(i\pi\tau) = \exp(-\pi\tau_2)$, it is clear that only relatively few terms will give the dominant contribution as $\beta \to \infty$. Using the explicit form of q and completing the squares in the exponent in (4.7) we find

$$\theta_1(z|\tau) = -ie^{\pi\tau_2 A^2} \sum_{n=-\infty}^{\infty} (-)^n e^{-\pi\tau_2 (n+1/2-A)^2} . \tag{4.8}$$

In the decompactification limit $\beta \to \infty$ the derivative of the theta function in the denominator of W_{eff} yields only minor contributions which combine with the exponent of the prefactor in (4.8) into a perfect square. We find that

$$W_{\text{eff}} = -2N\mu \left(1 + \mathcal{O}(e^{-2\pi\tau_2})\right) e^{\pi\tau_2 (A-1/2)^2} \sum_{n=-\infty}^{\infty} (-)^n e^{-\pi\tau_2 (n+1/2-A)^2} . \tag{4.9}$$

From this form it is not hard to recover the result derived before from the product presentation of theta functions. Thus, to evaluate the sum in (4.9) it is convenient to split A in its integer and fractional parts:

$$A = k + \alpha \qquad \text{with} \qquad k \in \mathbb{Z} \qquad \text{and} \qquad \alpha \in [0,1) . \tag{4.10}$$

Shifting the summation index by k it is easy to see that the remaining sum in W_{eff} is

$$\sum_{n=-\infty}^{\infty} (-)^n e^{-\pi\tau_2 (n+1/2-A)^2} = (-)^k \sum_{n=-\infty}^{\infty} (-)^n e^{-\pi\tau_2 (n+1/2-\alpha)^2} . \tag{4.11}$$

For any nonvanishing α this sum is dominated by the terms with $n = 0$ and $n = -1$. Since these two terms also lead to a vanishing W_{eff} for vanishing α, it is justified to define the effective superpotential in the decompactification limit to be

$$\lim_{\beta\to\infty} W_{\text{eff}} = (-1)^{k+1} 2N\mu e^{\pi\tau_2 (A-1/2)^2} \left(e^{-\pi\tau_2 (1/2-\alpha)^2} - e^{-\pi\tau_2 (1/2+\alpha)^2}\right) . \tag{4.12}$$

This equation captures all phenomena occurring in the family of 5 dimensional $\mathcal{N} = 1^*$ theories parametrized by g_5^2 and m_0 which are accessible from the submanifold of their Coulomb branch parametrized by the expectation value of $\cosh \beta\Phi_3$. In the next section we will analyze in detail its consequences and recover various features predicted by the supergravity analysis.

JHEP11(2003)001

5. Reconstructing the supergravity/string theory predictions for the moduli space

As we have seen in the previous section, the DV approach provides a description in terms of the superpotential (4.12) for the 5-dimensional $\mathcal{N}=1^*$ theory compactified on a circle of radius β with a nonvanishing $\mathcal{N}=2\to\mathcal{N}=1$ supersymmetry breaking term μ. This superpotential diverges as $\beta\to\infty$ and goes to zero when $\mu\to 0$. Thus, at first glance it seems as the only quantities that one could compare with the supergravity result must be independent of μ. Nevertheless, it is also possible to obtain meaningful quantities by taking double scaling limits $\beta\to\infty$ with $\mu\to 0$ such that some physical quantities remain constant.

In this section we use this strategy to show that the vacuum structure of the $\mathcal{N}=1^*$ theories changes drastically when $A\in\mathbb{Z}$, as predicted by the supergravity/string theory dual. The absence of polarization branches for $A\notin\mathbb{Z}$ and their presence when A is an integer emerges beautifully from the matrix model effective superpotential (4.12).

In section 6 we will present more evidence that the matrix model shows that new massless degrees of freedom appear when the theory is on a polarization branch, and will compare these predictions to the supergravity results and to the results of a perturbative computation. This hints toward the identification of the $A\in\mathbb{Z}$ theories with the RG fixed points conjectured by Seiberg in [21]. These points were further analyzed from field theoretic and geometric engineering standpoints in [24]–[27].

5.1 The polarization branches

Inspired by the supergravity description of the theory, it is natural to expect the polarization branch of the theory to be parameterized by the radius of the D4/NS5 configuration, which corresponds in the field theory to the expectation value of the sum of squares of transverse scalars. In our case, a measure of the radius of this configuration is

$$\frac{\beta}{g_5^2}\rho^2 \equiv \frac{\beta}{g_5^2}\langle \mathrm{Tr}[\Phi_1\Phi_2]\rangle = -\frac{\partial W}{\partial m_0}\,, \tag{5.1}$$

where the factor of β on the left hand side is consistent with the theory being the reduction of a 5 dimensional theory, m_0 is the 5 dimensional mass parameter[6] and ρ is a 4 dimensional chiral superfield. Note that the real part of ρ^2 is $\langle \mathrm{Tr}[X_5^2-X_6^2+X_8^2-X_9^2]\rangle$. Thus, a positive ρ^2 corresponds to polarization in the (5, 8) plane, while a negative $\langle\rho^2\rangle$ corresponds to polarization in the (6, 9) plane.

The key observation is that the 5-dimensional superpotential is highly constrained by the 8 supercharges, and can only have hypermultiplet mass terms or/and Yukawa couplings of the vector and hypermultiplets. Hence, the value of the superpotential at the critical points vanishes. Therefore, only the 4-dimensional vacua in which the value of the superpotential vanishes in the decompactification limit will become vacua of the 5-dimensional theory.

[6] This factor is also consistent with the normalization of the supersymmetry breaking deformation, since in the limit of small β this deformation becomes an ordinary mass term, with the same normalization as above.

JHEP11(2003)001

It is now fairly easy to obtain the moduli space. Since the effective superpotential depends on two parameters, μ and β, one can recover the decompactified $\mathcal{N} = 1^*$ theory by taking a double scaling limit $\mu \to 0$, $\beta \to \infty$. Every double scaling limit in which the superpotential vanishes will fix us somewhere on the moduli space. This will allow us to map a part of this moduli space which includes the polarization branch emanating from the origin of the Coulomb branch.

The analysis naturally splits into two different regimes. First, we consider the case $A \in \mathbb{Z}$, in which the effective superpotential vanishes for arbitrary values of μ and β and then we discuss the regime in which the effective superpotential vanishes only in the double scaling limit.

In the first case, even though the effective superpotential vanishes identically, its derivative is nonzero. More concretely, the expectation value of the polarization modulus is given by

$$\langle \rho^2 \rangle = 2 g_5^2 N \left[\mu e^{\frac{1}{2}\beta m_0 (A-1)} \right] . \tag{5.2}$$

Except for the case $A = 1$ which is not covered by this analysis, one can take double scaling limits $\mu \to 0$, $\beta \to \infty$ in which $\langle \rho^2 \rangle$ can be fixed to arbitrary, continuous values. Thus, the 5 dimensional theory has a continuum of vacua parameterized by the expectation value of ρ^2, which are exactly the vacua dual to polarized configurations of arbitrary radius. Furthermore, since the parameter μ is complex, the family of vacua is two-dimensional, matching the supergravity prediction.

Before proceeding, let us stress that capturing the polarization vacua through the DV approach depends on the choice of the $\mathcal{N} = 2 \to \mathcal{N} = 1$ breaking deformation. If we had chosen another type of deformation, we could have conceivably obtained only $\mathcal{N} = 1$ vacua which lift to vacua on the Coulomb branch, and would have remained oblivious to the existence of a polarization branch. Even though the perturbation we chose only maps the polarization branch for $A \geq 2$, this branch exists also for $A = 1$, and the investigation of the new degrees freedom we present in the next chapter is unchanged in this case. However, the double scaling limit of (5.2) does not allow us to access this branch.

Let us now turn to the second case, $A \notin \mathbb{Z}$. In this situation the double scaling limit is more constrained, since to recover an $\mathcal{N} = 2$ vacuum we need $W_{\text{eff}} \to 0$. Even if the algebra is more complicated than for integer A, extracting the large β scaling is straightforward, if one notices that the mass parameter m_0 appears only in the combination $\exp(-\beta m_0)$. Thus, it is clear that as far as the large β scaling is concerned, the would-be polarization modulus behaves as

$$\frac{\beta}{g_5^2} \langle \rho^2 \rangle = \frac{\partial W}{\partial m_0} \sim \beta W_{\text{eff}} \qquad \Rightarrow \qquad \langle \rho^2 \rangle \sim W_{\text{eff}} . \tag{5.3}$$

Since W_{eff} vanishes in the decompactification limit, the only vacuum allowed in the case of noninteger A is the one with vanishing expectation value for the polarization modulus. Hence, the field describing fluctuations in this radial direction is massive. This recovers the remaining part of the supergravity prediction about the structure of the space of $\mathcal{N} = 1^*$ theories – for non-integral A the branes are forced to sit at the origin of four of the five transverse directions.

JHEP11(2003)001

With hindsight, it is clear that the construction described above is the only possibility of reconstructing the superpotential while the usual integrating in procedure of [28] (see e.g. [29] for its application in the matrix model context) is doomed to fail since it is not possible to integrate in massless fields.

6. The new degrees of freedom of the fixed point theories

In this section we explore the new degrees of freedom which appear in the theory at integer A. We first discuss the supergravity predictions for the physics of the polarization branch and then examine the way in which the new degrees of freedom appear in the DV description. Though we will not be able to explicitly identify some of the light fields on the gauge theory side, we will point out some features which are consistent with their existence. We conclude with a perturbative computation, along the lines of [23, 21], which points to the identification of the $A \in \mathbb{Z}$ theories with the fixed points of the renormalization group conjectured in [21].

6.1 Supergravity predictions

In order to understand the physics of these new vacua, it is useful to examine the objects of the maximally supersymmetric theory living on parallel D4 branes. This theory has W-bosons and "W-strings", corresponding to F-strings and D2 branes stretched between the D4 branes. The theory also has a massive monopole (instanton), which is a D0 brane. When the theory is on the Coulomb branch, the W-bosons and W-strings acquire mass by the Higgs mechanism. The mass of the monopole is unchanged.

The objects in the perturbed theory are the same, but their properties change drastically on the polarization branch. Since near an NS5 brane the dilaton diverges, the masses of the D0 and D2 brane vanish and thus the theory has massless monopoles and massless "W-strings". The mass of the W-boson is proportional to the distance between the would-be location of D4 branes, which is proportional to the radius of the polarized configuration. Thus, at a generic point on the polarization branch, the W-bosons are massive. However, as one approaches the origin of the polarization branch along the polarization branch, the mass of these W-bosons also goes to zero.

Hence, the W-bosons and W-strings, which couple electrically and magnetically to the gauge field on the brane, are massless; the other objects in the theory are massless as well. This suggests that approaching the origin from the polarization branch, brings us to a nontrivial fixed point of the renormalization group flow that bears certain similarities to Argyres-Douglas type theories in four dimensions [30].[7] The fact that all the objects of this theory become massless also suggests that, much like in the Argyres-Douglas case, the theory becomes superconformal.[8]

[7]It is important to note that approaching the origin of the polarization branch from the Coulomb branch does not give the same point. Indeed, the monopoles (D0 branes) are never massless on the Coulomb branch. Moreover, when the branes are on the Coulomb branch supergravity is never weakly coupled in their vicinity. Therefore, the IR physics is described by the field theory, and we cannot use the string descriptions of W-bosons and W-strings to conclude they are massless.

[8]It is however not obvious how to compute the β function and show that it vanishes.

It is perhaps an appropriate moment to return to the observation made in section 2 that the area of the horizon of the near extremal solution (2.4) is independent of the deformation parameter B for fixed temperature. This implies that, for fixed temperature, the entropy of the field theory is independent on the mass of the hypermultiplet. This might seem to contradict the fact that by varying m it is possible to reach points where the theory has extra light degrees of freedom.

This puzzle is solved by observing that these light degrees of freedom are nonperturbative, and thus only visible in supergravity. As the solution is taken off extremality, only the origin of the Coulomb branch survives as a possible vacuum of the theory. Both the Coulomb branch and the polarization branch are lifted by the temperature deformation. However, as pointed out above, the origin does not belong to the polarization branch and thus the extra massless degrees of freedom are not part of the field theory excitation spectrum.

6.2 The effective coupling

As we have seen in the previous section, to recover the 5-dimensional theory we are interested in, we must take the double scaling limit $\beta \to \infty$, $\mu \to 0$ with $W_{\text{eff}} \to 0$. However, some of the quantities of this theory are independent of μ, which makes them easy to obtain and study even before taking this limit.

One such quantity is the effective gauge coupling. The relation between 4 dimensional $\mathcal{N} = 1$ gauge theories and special geometry [6] implies that this coupling is independent of the overall scale of the tree level superpotential. This property survives also in the present case, as it can be easily seen from equation (3.4). Thus, if

$$2\pi i \ \tau_{\text{eff}} = \frac{\partial^2 F}{\partial S^2} \tag{6.1}$$

is well-defined, then it is also independent of μ. Explicitly computing the effective coupling requires one to (at least in principle) be able to invert the equation $S = S(t)$. This is only the case if $h''(t) \neq 0$ since

$$\frac{\partial^2 F}{\partial t \, \partial S} = t h''(t) \qquad \frac{\partial S}{\partial t} = \frac{1}{2\pi i} h''(t) \, . \tag{6.2}$$

Thus, in the case of the nondegenerate solutions of (3.7) the effective coupling is constant and given by the 't Hooft coupling

$$\tau_{\text{eff}} = \frac{\tau}{N} \, . \tag{6.3}$$

Before we proceed to the degenerate case, let us note that this seems quite different from the effective coupling computed in [8], or the one we will compute in section 6.2 using perturbative arguments.

The difference arises from the fact that the the supersymmetry breaking deformation (3.3) selects a vacuum which is at finite distance from the origin of the Coulomb branch. Even though the superpotential (3.8) does not allow us to explicitly compute $\langle \mathrm{Tr}[\Phi_3^{2k}] \rangle$ for all k, we can still estimate some of them in the decompactification limit.

From equation (4.12)[9] we find

$$\langle \mathrm{Tr}[\Phi_3^2] \rangle^{\frac{1}{2}} \sim 2mA \qquad \text{for} \qquad A \notin \mathbb{Z}\,, \tag{6.4}$$

and similar expressions for the other invariants. Thus, although we have started from a configuration where Φ_3 is zero (which was reflected in the choice for the matrix model saddle point), the multilinears in this field acquire nontrivial expectation values at the quantum level.

Equation (6.3) is also consistent with the supergravity description of the $\mathcal{N} = 1^*$ theories. Indeed, we can show that by reducing to 4 dimensions the vacua selected by the supersymmetry breaking term in (3.3), we find (up to possible T dualities) a background where the dilaton near the branes is independent of m.

From (6.4) it is easy to see that the size of the D-brane distribution dual to the selected Coulomb branch configuration is of order $mA \sim m^2 g_s N$. For this configuration, supergravity is a valid description for $m \lesssim r \lesssim \frac{1}{m}$, which for a fixed A translates into

$$\frac{1}{g_5^2 N} \lesssim r \lesssim g_5^2 N\,. \tag{6.5}$$

i.e. the supergravity description is valid almost in the entire space.

For a distribution of this size, close enough to the branes, the two terms defining the dilaton in (2.2) are of the same size. Therefore, near the branes the dilaton behaves as

$$\frac{e^{4\phi/3}}{g_s^{4/3}} = \Lambda = Z^{-1/3}\,(1+\alpha)\,, \tag{6.6}$$

where α is a constant independent of m. The metric coefficients have a similar behavior. In particular, the coefficient corresponding to the new compactified direction is proportional to $Z^{-1/2}$, up to a constant independent of m. These are (up to some irrelevant numerical coefficients) precisely the characteristics of an unperturbed D4 brane solution. It is then clear that by T-dualizing along the circle the dilaton becomes m-independent, as predicted by the DV analysis (6.3).

Let us now return to the case of the degenerate solutions in (3.7). Since the relation between S and t cannot be inverted, the effective coupling τ_{eff} is undetermined. Moreover, in the decompactification limit, the nondegenerate vacua of theories with $A \in \mathbb{Z}$ asymptote to degenerate vacua.[10]

Therefore, for integer A, τ_{eff} is undetermined by (6.1). These features suggest a breakdown of the low energy effective theory which can occur due to the appearance of new light degrees of freedom. This interpretation is consistent with the supergravity analysis discussed in the previous subsection.

The effective coupling τ_{eff} is related to the expectation value of the bulk dilaton, which varies as one changes the polarization radius. Therefore the value of τ_{eff}, though undetermined by gauge theory, can be used to parametrize (at least locally) the moduli

[9] Using $\langle \mathrm{Tr}(\cosh(\beta\Phi_3) - 1) \rangle = \frac{\partial W_{\text{eff}}}{\partial \mu}$

[10] One can easily see this by noticing that the points where $h(t) = 0$ and $h''(t) = 0$ are separated by $\delta t \sim \frac{1}{\beta}$.

JHEP11(2003)001

space of vacua. The degeneracy discussed above is therefore a reflection of the fact that to one classical vacuum corresponds a continuum of quantum vacua, indistinguishable in the particular matrix model used here.

To summarize, when $A \notin \mathbb{Z}$ the effective coupling is well-defined and proportional to the bare coupling at all points on the Coulomb branch selected by the $\mathcal{N}=2 \to \mathcal{N}=1$ deformation term (3.3). However, when $A \in \mathbb{Z}$, $\tau_{\rm eff}$ is undetermined and the expression of the glueball superfield S is degenerate, suggesting a breakdown of the low energy effective description and the appearance of new light degrees of freedom.

6.3 Perturbative breakdown

Some of the special points singled out both by the supergravity and the DV description of the 5 dimensional $\mathcal{N}=1^*$ theory can also be identified perturbatively, using the techniques of [21] and [23]. It is not hard to compute the effective coupling of 5-dimensional theories since, as argued in [21], it is only corrected at one loop. These corrections lead to the vanishing of the effective τ parameter, and imply the existence of a regime of ultra-strong coupling. This will occur at the first nontrivial points in the series of theories obeying

$$A \in \mathbb{Z}\,. \tag{6.7}$$

Since the effective coupling constant is related to a 5 dimensional Chern-Simons term, it receives a finite shift at 1 loop. According to [21, 23], in a theory containing hypermultiplets of masses m_i, this correction has the general form

$$\frac{1}{g_{\rm eff}^2} = \frac{1}{g_5^2} + a\phi - \sum_i b_i|\phi - m_i|\,, \tag{6.8}$$

where ϕ is the classical expectation value of the scalar in the vector multiplet. The first term on the right hand side is the bare gauge coupling , and the following ones are due to hypermultiplet interactions. The coefficients b_i depend on the representation of the i^{th} hypermultiplet. It is easy to see that these terms arise at the 1-loop level from the self-energy diagram of the vector multiplet, and therefore their coefficients are equal to $\mathrm{Tr}_{R_i}[T^aT^b]$ in the appropriate representation R_i, up to a universal coefficient.

To match the assumptions which led to (3.8), we are interested in the configuration at the origin of the Coulomb branch (where the gauge group is classically unbroken) which corresponds to $\phi = 0$. In this case only the hypermultiplets contribute; furthermore, since there exists only one hypermultiplet (of mass m, transforming in the adjoint representation), the 1 loop effective coupling is given by

$$\frac{1}{g_{\rm eff}^2} = \frac{1}{g_5^2} - \frac{1}{4\pi^2}N|m|\,. \tag{6.9}$$

Consequently, the effective 5-dimensional coupling vanishes for

$$g_5^2 N|m| = 4\pi^2 \tag{6.10}$$

which is the first case in the series (6.7).

In [21] such points were given the interpretation of fixed points of the renormalization group, exhibiting enhanced symmetries. From the supergravity analysis it is certainly the case that a number of mutually nonlocal massless states appear at these points. From that analysis, it is however not clear what is the enhanced global symmetry. Clearly, it would be interesting to investigate this further.

7. Conclusions and future directions

We have compared the supergravity/string theory and the matrix model descriptions of the $\mathcal{N} = 1^*$ theory in 5 dimensions, and have found that the unusual properties displayed by these theories for $\frac{1}{4\pi^2} g_5^2 Nm \in \mathbb{Z}$ are captured by both approaches.

The existence of a nonperturbative branch of the moduli space, which in the supergravity corresponds to the possibility of polarizing D4 branes into NS5 branes, emerges in the matrix model from certain properties of Jacobi θ functions. The theories along the polarization branch posses new light degrees of freedom. At the supergravity level they are the polarization modulus as well as D0 and D2 branes which become massless near the polarized D4 branes. In the matrix model description, these new degrees of freedom manifest themselves through the breakdown of the effective description at the special points $\frac{1}{4\pi^2} g_5^2 Nm \in \mathbb{Z}$.

We have also compared the matrix model description of the 5-dimensional $\mathcal{N} = 1^*$ theory to the integrable models-based Seiberg-Witten-like description [8], as well as with perturbative arguments along the lines of [21, 23]. These 4 descriptions agree with each other in the ranges of mutual validity.

The supergravity description allowed us to prove a conjecture of Nekrasov that a certain shift of the hypermultiplet mass is a symmetry of this theory. We have also computed the entropy of this theory, and found that at finite temperature it is independent of m, which is again consistent with a shift in m being a symmetry of the theory. The picture which emerges is that the spectrum of the theory is reshuffled under this transformation.

Perhaps the most intriguing question raised by our analysis is *why* are the theories with $A \in \mathbb{Z}$ special. Except the possibility for brane polarization, the supergravity duals show no special features at these particular points. On the gauge theory side one might contemplate a 6-dimensional interpretation.[11] The lift to six dimensions of the $\mathcal{N} = 1^*$ theory is the (2,0) theory perturbed with a vector potential which breaks 6-dimensional Lorentz invariance. This theory and its gravity dual have been studied in [31], and no indication of any special points has emerged. It is possible that a DLCQ description of these theories might reveal aspects of the special physics associated with these points. However, there is no straightforward 6 dimensional (2,0) explanation of *why* these points are special. A source of the difficulties can be traced to the fact that the 5 dimensional coupling emerges only in the compactification process and has no interpretation in the strict six dimensional limit.

[11] A loosely similar example with these characteristics was discussed in [32] where it was shown that the partition function of string theory on a Melvin background exhibits certain equally-spaced zeroes. The analogous analysis in the case at hand is unfortunately prohibitive.

The setup discussed in this paper can be generalized by adding of other branes, orbifolds and orientifolds, to include other gauge groups and matter representations. The perturbative arguments described in section 6.3 apply rather straightforwardly to these theories and suggest that, at least in some cases, they should also have an infinite set of points where exceptional physics emerges. This could also be a hint that such theories have a shift symmetry similar to the one studied here, while the arguments of [21] would suggest that these points are fixed under RG flow.

These two possibilities could be ascertained in three ways: by providing a supergravity dual to these theories (which is probably straightforward for the orbifold/orientifold case) by solving the quantum mechanics with $\mathrm{SO}(N)$, $\mathrm{Sp}(N)$ gauge groups and different matter representations or by solving the integrable system analogous to the one in [8]. It is however not *a priori* clear which approach is the most profitable one.

It would also be interesting to solve the matrix quantum mechanics for a generic $\mathcal{N} = 2 \to \mathcal{N} = 1$ deformation, and thus map the full moduli space of the 5-dimensional $\mathcal{N} = 1^*$ theory. This would allow one to confirm the supergravity prediction that the Coulomb branch intersects the polarization branch at all points of unbroken gauge symmetry, and to study the way one moves between these two branches.

Another interesting direction to explore is the limiting behavior of the theories along the polarization branch as the radius of polarization ($\langle \mathrm{Tr}[\Phi_1\Phi_2]\rangle$) goes to infinity. In this limit the circular D4-NS5 brane configuration decompactifies. Although ten dimensional supergravity might not valid in this regime (unless one also scales N with the radius), the field theory analysis indicates that such a limit makes sense. It would be interesting to study the theory in this limit and to determine whether in this case it is possible to find a 6-dimensional explanation of the exceptional 5-dimensional physics.

Many of the arguments which lead to the phenomena we have discovered for the $\mathcal{N} = 1^*$ theory in 5-dimensions seem to generalize to other 5-dimensional theories. This may lead one to believe that nonperturbative branches of the moduli space and mass shift periodicities are generic features of field theories in 5 dimensions. Understanding whether this is indeed the case, and if so why, is an important open question.

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JHEP11(2003)001

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